

THE LATTICE OF l -GROUP VARIETIES

BY

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ABSTRACT. For any type of abstract algebra, a variety is an equationally defined class of such algebras. Recently, attempts have been made to study varieties of lattice-ordered groups (l -groups). Martinez has shown that the set L of all l -group varieties forms a lattice under set inclusion with a compatible associative multiplication. Certain varieties \mathcal{S}_p (p prime) have been proved by Scrimger to be minimal nonabelian varieties in L . In the present paper, it is shown that these varieties can be used to produce varieties minimal with respect to properly containing various other varieties in L . Also discussed are the relations among the \mathcal{S}_n ($n \in N$), and it is established that all infinite collections of the \mathcal{S}_n have the same least upper bound in L . Martinez has also classified l -groups using torsion classes, a generalization of the idea of varieties. It is proved here that L is not a sublattice of T , the lattice of torsion classes.

1. Introduction. As the study of varieties of groups has proved to be such a useful tool in group theory, it is natural that varieties of lattice-ordered groups (l -groups) are of interest to those concerned with l -group theory. The first works to appear in this area dealt mainly with specific varieties. Weinberg's work on abelian l -groups [14] led to the conclusion that the abelian variety \mathcal{Q} is the smallest nontrivial l -group variety. Martinez [9] considered a family of varieties \mathcal{L}_n ($n \in N$) defined by the equation $x^n y^n = y^n x^n$. Scrimger [13], working within the \mathcal{L}_n showed that certain varieties $\mathcal{S}_p \subseteq \mathcal{L}_p$ cover \mathcal{Q} for all primes p . Holland [6] proved that the variety \mathcal{U} of normal-valued l -groups contains every proper l -group variety.

A more comprehensive study of l -group varieties was made by Martinez [10], in which the set L of all l -group varieties was shown to form a complete lattice, using containment as the partial order, with a compatible multiplication. The present paper is an attempt to extend the work on the structure of L and to further investigate the role of the \mathcal{S}_n 's within this lattice. The notation to be used will follow the lines of that used by H. Neumann [12].

Starting near the top of the lattice L , we show in (2.2) that \mathcal{U} is finitely join-irreducible. At the other extreme of the lattice, it is first shown in (3.11) and (3.12) that the sublattice of L generated by $\{\mathcal{S}_p | p \text{ is prime}\}$ is lattice-isomorphic to the lattice of finite sets of primes. However, this isomorphism does not extend to the complete sublattice generated by $\{\mathcal{S}_p | p \text{ is prime}\}$, as it is shown in (3.14) and (3.17) that the join of any infinite collection of \mathcal{S}_n 's is the variety generated by $Z \text{ Wr } Z$.

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The other main result of this paper deals with the relationship between \mathbf{L} and \mathbf{T} , the set of all torsion classes of l -groups. It has been shown by Martinez [11] that \mathbf{T} also forms a lattice and a semigroup. Holland [7] has shown that $\mathbf{L} \subseteq \mathbf{T}$. As both the meet and multiplication operations are identical in \mathbf{L} and \mathbf{T} , it is natural to ask whether \mathbf{L} forms a sublattice of \mathbf{T} . In (4.3) and (4.4) it is shown that this is not the case and that in general the join operations of \mathbf{L} and \mathbf{T} do not yield the same results.

2. The lattice-ordered semigroup \mathbf{L} . A *variety* of l -groups is the class of all l -groups for which a given set of words (involving the group and lattice operations) are laws. Equivalently a variety of l -groups is a class of l -groups which is closed with respect to l -subgroups, l -homomorphic images, and cardinal products (direct products with componentwise ordering). Some of the l -group varieties of particular interest are \mathfrak{E} the trivial variety, \mathfrak{A} the abelian variety, \mathfrak{R} the representable variety, \mathfrak{N} the normal-valued variety, and \mathfrak{L} the variety of all l -groups.

The set \mathbf{L} of all l -group varieties is known to be a dually-Brouwer and hence distributive lattice, with intersection functioning as the meet operation, and the join of a collection of varieties defined to be the intersection of all varieties containing those in the collection. Similarly, it is possible to describe the variety generated by an l -group G , $l\text{-var}(G)$, as the intersection of all l -group varieties containing G . In addition, we may define an associative operation on \mathbf{L} such that for $\mathfrak{U}, \mathfrak{V} \in \mathbf{L}$, $\mathfrak{U} \mathfrak{V}$ contains all l -groups G having an l -ideal in \mathfrak{U} whose quotient lies in \mathfrak{V} . This operation is compatible with the set containment ordering on \mathbf{L} and distributes over all meets and finite joins from the left.

Martinez [10] has shown that $G \in \mathfrak{V}_1 \vee \mathfrak{V}_2$ iff G is a subdirect product of l -groups from \mathfrak{V}_1 and \mathfrak{V}_2 . He has also shown that \mathfrak{N} is idempotent in \mathbf{L} , i.e. $\mathfrak{N}\mathfrak{N} = \mathfrak{N}$. It is now to be shown that \mathfrak{N} is finitely join-irreducible as well; however, the proof of this fact will require the use of wreath products.

Let G and H be l -groups. Due to a theorem of Holland [5], there exist chains Ω_1 and Ω_2 such that G and H can be considered as l -groups of order-preserving permutations (l -permutation groups) on Ω_1 and Ω_2 respectively. It should be noted that an order-preserving permutation g on a chain Γ is considered positive if $\gamma g > \gamma$ for all $\gamma \in \Gamma$. Using this representation for G and H , let $(W, \Omega) = (G, \Omega_1) \text{Wr} (H, \Omega_2)$ be the order-preserving permutation group on $\Omega = \Omega_1 \tilde{\times} \Omega_2$, where all elements of W are of the form (θ, h) for $h \in H$ and $\theta: \Omega_2 \rightarrow G$, and the action of W on Ω defined by $(\alpha_1, \alpha_2)(\theta, h) = (\alpha_1\theta(\alpha_2), \alpha_2h)$ for any $(\alpha_1, \alpha_2) \in \Omega$. It can be verified that the group operation on W is such that $(\theta, h)(\phi, k) = (\psi, hk)$, where for $\alpha \in \Omega_2$, $\psi(\alpha) = \theta(\alpha)\phi(\alpha h)$. Using this multiplication on W and the usual ordering of order-preserving permutations discussed above, W is an l -group [8], and is called the Wreath product of G with H . Within W ,

$$W' = \{(\theta, h) | \theta(\alpha) = e_G \text{ for all but a finite subset of } \Omega_2\}$$

forms an l -subgroup which also acts as an l -permutation group on $\Omega = \Omega_1 \tilde{\times} \Omega_2$. This l -subgroup is called the wreath product of G with H and is denoted by $(W', \Omega) = (G, \Omega_1) \text{wr} (H, \Omega_2)$. When an l -group G is itself totally ordered, it is

possible to consider G as an l -permutation group on itself, using right translation as the action. In such a case, G is usually preferred over the more cumbersome notation (G, G) ; so the notation $(G, \Omega_1) \text{ Wr } H$, $G \text{ Wr } (H, \Omega_2)$, $G \text{ Wr } H$, etc. make sense in this context and will be used where appropriate.

2.1. LEMMA. $\mathcal{V} \in \mathbf{L}$ is finitely join-irreducible if \mathcal{V} is closed under wreath (Wreath) products.

PROOF. By way of contradiction, suppose \mathcal{V} is finitely join-reducible. Then there exist $\mathcal{U}_1, \mathcal{U}_2 \in \mathbf{L}$ such that $\mathcal{U}_1 \subset \mathcal{V}$, $\mathcal{U}_2 \subset \mathcal{V}$ and $\mathcal{V} = \mathcal{U}_1 \vee \mathcal{U}_2$. Since a variety is generated by its subdirectly irreducible l -groups [4], there must exist subdirectly irreducible l -groups $H \in \mathcal{V} \setminus \mathcal{U}_1$ and $K \in \mathcal{V} \setminus \mathcal{U}_2$. Holland [5] has shown that there exist chains Ω_1 and Ω_2 such that (H, Ω_1) and (K, Ω_2) are transitive l -permutation groups, i.e. a group G acts transitively on a set Γ if for all $\gamma, \delta \in \Gamma$ there exists $g \in G$ such that $\gamma g = \delta$. Then $(W, \Omega) = (H, \Omega_1) \text{ wr } (K, \Omega_2)$ is also a transitive l -permutation group (similarly for the Wreath product), and $W \in \mathcal{V}$ by assumption. Thus, $W \in \mathcal{U}_1 \vee \mathcal{U}_2$, so W must be a subdirect product of l -groups from \mathcal{U}_1 and \mathcal{U}_2 , i.e. there must exist l -ideals A_1 and A_2 in W such that $A_1 \cap A_2 = \{e\}$, $W/A_1 \in \mathcal{U}_1$, and $W/A_2 \in \mathcal{U}_2$. If A_1 were trivial, then $W \in \mathcal{U}_1$; and since H is l -embeddable in W , then $H \in \mathcal{U}_1$, contradicting our selection of H . If A_2 were trivial, then $W \in \mathcal{U}_2$; and since K is an l -homomorphic image of W , $K \in \mathcal{U}_2$, contradicting our selection of K . Thus, neither A_1 nor A_2 can be trivial. Hence, there must exist $h_1 \in A_1^+$, $h_2 \in A_2^+$, and $\alpha, \beta \in \Omega$ such that $\alpha h_1 > \alpha$ and $\beta h_2 > \beta$. Also, by transitivity, there exists $g \in W$ such that $\alpha g = \beta$. Now since $\beta g^{-1} h_1 g = \alpha h_1 g > \alpha g = \beta$, both $g^{-1} h_1 g \in A_1$ and $h_2 \in A_2$ move β . But this is impossible for disjoint l -ideals. Therefore, \mathcal{V} must be finitely join-irreducible.

2.2. THEOREM. \mathcal{U} is finitely join-irreducible in \mathbf{L} .

PROOF. Martinez [10] has shown that \mathcal{U} is idempotent, and hence closed with respect to all extensions including Wreath and wreath products. Thus, by (2.1) \mathcal{U} is finitely join-irreducible.

3. Properties of the Scrimger varieties. It is known that \mathcal{Q} is the smallest nontrivial variety in \mathbf{L} . In an attempt to generalize \mathcal{Q} , the family of varieties \mathcal{L}_n for $n \in N$ were introduced, where \mathcal{L}_n is the variety of all l -groups in which all n th powers commute. Obviously, $\mathcal{L}_1 = \mathcal{Q}$ and $\mathcal{L}_m \subseteq \mathcal{L}_n$ if $m|n$. Also, within each \mathcal{L}_n is an l -group [9] of the form

$$G_n = \{(\phi, k) \in Z \text{ Wr } Z \mid \phi(i) = \phi(j) \text{ if } i \equiv j \pmod{n}\}.$$

Such an l -group may also be viewed as a lexicographically-ordered splitting extension by Z of a cardinal product of n copies of Z , with the multiplication defined by

$$\begin{aligned} ((r_0, \dots, r_{n-1}), k) & ((s_0, \dots, s_{n-1}), l) \\ &= ((r_0 + s_0 + k, \dots, r_{n-1} + s_{n-1} + k), k + l), \end{aligned}$$

where $r_i, s_j, k, l \in \mathbb{Z}$ and all subscripts are considered mod n . These l -groups and the varieties they generate, where $\mathcal{S}_n = l\text{-var}(G_n)$ is known as the n th Scrimger variety, have been shown to be of great interest in the study of \mathbf{L} . Some of the more useful facts concerning the G_n and \mathcal{S}_n are:

- 3.1. For any $n \in N$, G_n is an l -subgroup of $Z \text{ Wr } Z$.
- 3.2. For any $n \in N$, G_n is a subdirectly irreducible l -group.
- 3.3. For any $n \in N$, $G_n \in \mathcal{Q}^2 = \mathcal{Q}\mathcal{Q}$.
- 3.4. For $m, n \in N$, $G_m \in \mathcal{L}_n$ iff $m|n$.
- 3.5. For $m, n \in N$, $\mathcal{L}_m \subseteq \mathcal{L}_n$ iff $m|n$.
- 3.6. For $m, n > 1$, $\mathcal{S}_m \vee \mathcal{S}_n \subset \mathcal{S}_{mn}$ and $\mathcal{L}_m \vee \mathcal{L}_n \subset \mathcal{L}_{mn}$.
- 3.7. For any $n \in N$, $\mathcal{R} \wedge \mathcal{L}_n = \mathcal{Q}$ and $\mathcal{R} \wedge \mathcal{S}_n = \mathcal{Q}$ [9].
- 3.8. For $m, n \in N$ such that $(m, n) = 1$, $\mathcal{L}_m \wedge \mathcal{L}_n = \mathcal{Q}$, $\mathcal{S}_m \wedge \mathcal{S}_n = \mathcal{Q}$, and $\mathcal{L}_m \wedge \mathcal{S}_n = \mathcal{Q}$ [13].
- 3.9. For any prime p , \mathcal{S}_p covers \mathcal{Q} in \mathbf{L} , i.e. no variety lies between \mathcal{Q} and \mathcal{S}_p [13].

3.10. PROPOSITION. For any variety $\mathcal{V} \in \mathbf{L}$ and any prime p such that $\mathcal{V} \wedge \mathcal{S}_p = \mathcal{Q}$, $\mathcal{V} \vee \mathcal{S}_p$ covers \mathcal{V} in \mathbf{L} .

PROOF. Since \mathbf{L} is a distributive lattice, the interval $\{\mathcal{W} \in \mathbf{L} | \mathcal{V} \subset \mathcal{W} \subset \mathcal{V} \vee \mathcal{S}_p\}$ is lattice-isomorphic to the interval $\{\mathcal{U} \in \mathbf{L} | \mathcal{Q} = \mathcal{V} \wedge \mathcal{S}_p \subset \mathcal{U} \subset \mathcal{S}_p\}$ [3], and hence is empty.

Thus, it is known that for any prime p , $\mathcal{R} \vee \mathcal{S}_p$ covers \mathcal{R} . Also, for any integer n and prime p such that $p \nmid n$, $\mathcal{S}_n \vee \mathcal{S}_p$ covers \mathcal{S}_n and $\mathcal{L}_n \vee \mathcal{S}_p$ covers \mathcal{L}_n . Extending this notion by means of induction leads us to the following result.

3.11. COROLLARY. For any finite set of distinct primes $\{p_1, p_2, \dots, p_m\}$, $\bigvee \{\mathcal{S}_{p_i} | i = 1, \dots, m\}$ covers $\bigvee \{\mathcal{S}_{p_i} | i = 1, \dots, m-1\}$.

3.12. PROPOSITION. For any primes $p_1, \dots, p_m, q_1, \dots, q_n$,

$$\bigvee \{\mathcal{S}_{p_i} | i = 1, \dots, m\} \subseteq \bigvee \{\mathcal{S}_{q_j} | j = 1, \dots, n\}$$

iff $\{p_1, \dots, p_m\} \subseteq \{q_1, \dots, q_n\}$.

PROOF. Sufficiency is obvious. To prove necessity, suppose $\bigvee \{\mathcal{S}_{p_i} | i = 1, \dots, m\} \subseteq \bigvee \{\mathcal{S}_{q_j} | j = 1, \dots, n\}$. Then for any p_i ,

$$\mathcal{S}_{p_i} \subseteq \mathcal{S}_{p_i} \wedge \left(\bigvee \{\mathcal{S}_{q_j} | j = 1, \dots, n\} \right) = \bigvee \{(\mathcal{S}_{p_i} \wedge \mathcal{S}_{q_j}) | j = 1, \dots, n\}.$$

Hence, for some j , $\mathcal{S}_{p_i} = \mathcal{S}_{p_i} \wedge \mathcal{S}_{q_j}$, implying that $p_i = q_j$.

Accordingly, the sublattice of \mathbf{L} generated by $\{\mathcal{S}_p | p \text{ is prime}\}$ is lattice-isomorphic to the lattice of finite sets of primes. To investigate whether or not the complete sublattice of \mathbf{L} generated by $\{\mathcal{S}_p | p \text{ is prime}\}$ is lattice-isomorphic to the full Boolean lattice of sets of primes, it is helpful to consider conditions under which an l -group is l -isomorphic to G_n for some $n \in N$.

3.13. THEOREM. An l -group G is l -isomorphic to G_n iff it is generated (as a group) by two positive elements a and b such that

- (i) $a < b$, and
- (ii) for $1 \leq i \leq n-1$, $b^{-i}ab^i \wedge a = e$ and $b^{-n}ab^n = a$.

PROOF. To prove necessity, it need only be noted that the elements $((1, 0, \dots, 0), 0)$ and $((0, \dots, 0), 1)$ of S_n satisfy the conditions on a and b respectively.

To prove sufficiency, let us assume that G is an l -group generated by positive elements a and b satisfying conditions (i) and (ii). Define a function $\nu: G_n \rightarrow G$ by:

$$((r_0, \dots, r_{n-1}), k)\nu = \left(\prod_{i=0}^{n-1} (b^{-i}ab^i)^{r_i} \right) b^k,$$

where $r_i, k \in \mathbb{Z}$. To verify that ν preserves the group operations, it should be noted that condition (ii) guarantees that all conjugates of a by powers of b commute in G . So, for arbitrary elements of G_n ,

$$\begin{aligned} [((r_0, \dots, r_{n-1}), k)((s_0, \dots, s_{n-1}), l)]\nu \\ = [((r_0 + s_0 + k, \dots, r_{n-1} + s_{n-1} + k), k + l)]\nu \\ = \left(\prod_{i=0}^{n-1} (b^{-i}ab^i)^{r_i + s_i + k} \right) b^{k+l}, \end{aligned}$$

while

$$\begin{aligned} ((r_0, \dots, r_{n-1}), k)\nu((s_0, \dots, s_{n-1}), l)\nu \\ = \left(\prod_{i=0}^{n-1} (b^{-i}ab^i)^{r_i} \right) b^k \left(\prod_{j=0}^{n-1} (b^{-j}ab^j)^{s_j} \right) b^l \\ = \left(\prod_{i=0}^{n-1} (b^{-i}ab^i)^{r_i} \right) \left(\prod_{j=0}^{n-1} (b^{-j+k}ab^{j-k})^{s_j} \right) b^{k+l} \\ = \left(\prod_{i=0}^{n-1} (b^{-i}ab^i)^{r_i} \right) \left(\prod_{i=0}^{n-1} (b^{-i}ab^i)^{s_i+k} \right) b^{k+l} \\ = \left(\prod_{i=0}^{n-1} (b^{-i}ab^i)^{r_i + s_i + k} \right) b^{k+l}, \end{aligned}$$

with all subscripts considered mod n . Hence, ν is a group homomorphism.

In order to prove that ν preserves the lattice operations as well, it is helpful to first show that b exceeds the l -ideal of G generated by a . Suppose there exists $m \in \mathbb{N}$ such that $a^m \not\leq b$. If $a^m = b$, then a and b would commute. So, in fact, $a^m \not\leq b$, and then $(b^{-1}ab)^m \not\leq b$. By Holland's theorem [5], there exists a chain Γ such that G acts as an l -group of order-preserving permutations on Γ . Thus, there exists some $\gamma \in \Gamma$ for which

$$\gamma(b^{-1}ab)^m > \gamma b \geq \gamma. \quad (*)$$

Since γ is moved by $(b^{-1}ab)^m$, it is moved by $b^{-1}ab$. Then condition (ii) implies that $\gamma a = \gamma$, so $(\gamma b)(b^{-1}ab) = \gamma b$ and $(\gamma b)(b^{-1}ab)^m = \gamma b$. But by (*), $(\gamma b)(b^{-1}ab)^m > \gamma(b^{-1}ab)^m > \gamma b$. Thus $a^m < b$ for all $m \in \mathbb{Z}$, and similarly $(b^{-i}ab^i)^m < b$ for all $i, m \in \mathbb{Z}$. Now this and the disjointness of the conjugates leads us to:

$$\prod_{i=0}^{n-1} (b^{-i}ab^i)^{r_i} = \bigvee_{i=0}^{n-1} (b^{-i}ab^i)^{r_i} < b$$

for any $r_0, \dots, r_{n-1} \in \mathbb{Z}$. Again this inequality must in fact be proper, as a and b do not commute in G . Hence, b exceeds the l -ideal of G generated by a and so dominates the order on G .

Now for any $((r_0, \dots, r_{n-1}), k) \in G_n$, $((r_0, \dots, r_{n-1}), k) \vee ((0, \dots, 0), 0)$ is computed to be $((r_0, \dots, r_{n-1}), k)$ if $k > 0$, $((0, \dots, 0), 0)$ if $k < 0$, and $((r_0 \vee 0, \dots, r_{n-1} \vee 0), 0)$ if $k = 0$. In the case where $k > 0$,

$$\left(\prod_{i=0}^{n-1} (b^{-i}ab^i)^{r_i} \right) b^k \vee e = \left(\prod_{i=0}^{n-1} (b^{-i}ab^i)^{r_i} \right) b^k$$

since b^k dominates. Similarly, for $k < 0$,

$$\left(\prod_{i=0}^{n-1} (b^{-i}ab^i)^{r_i} \right) b^k \vee e = e.$$

For $k = 0$, let us divide $\{0, \dots, n-1\}$ into two disjoint sets $I = \{i | r_i > 0\}$ and $J = \{j | r_j < 0\}$. Then

$$\begin{aligned} \left(\prod_{i=0}^{n-1} (b^{-i}ab^i)^{r_i} \right) \vee e &= \left[\prod_{i \in I} (b^{-i}ab^i)^{r_i} \vee \prod_{j \in J} (b^{-j}ab^j)^{-r_j} \right] \prod_{j \in J} (b^{-j}ab^j)^{r_j} \\ &= \left[\prod_{i \in I} (b^{-i}ab^i)^{r_i} \prod_{j \in J} (b^{-j}ab^j)^{-r_j} \right] \prod_{j \in J} (b^{-j}ab^j)^{r_j} \\ &= \prod_{i \in I} (b^{-i}ab^i)^{r_i} = \prod_{i=0}^{n-1} (b^{-i}ab^i)^{r_i \vee 0}. \end{aligned}$$

Thus ν is an l -homomorphism.

Suppose $(\prod_{i=0}^{n-1} (b^{-i}ab^i)^{r_i}) b^k = e$. Then $k = 0$, since b strictly exceeds all products of conjugates of a . Now let us divide $\{0, \dots, n-1\}$ into I and J as above. Then $\prod_{i \in I} (b^{-i}ab^i)^{r_i} = \prod_{j \in J} (b^{-j}ab^j)^{-r_j}$. So if some $r_i > 0$, then

$$\begin{aligned} (b^{-i}ab^i)^{r_i} &= (b^{-i}ab^i)^{r_i} \wedge \prod_{j \in J} (b^{-j}ab^j)^{-r_j} \\ &= \prod_{j \in J} ((b^{-i}ab^i)^{r_i} \wedge (b^{-j}ab^j)^{-r_j}) = e, \end{aligned}$$

contradicting the fact that $b^{-i}ab^i > e$. Thus $r_0 = r_1 = \dots = r_{n-1} = 0$ and ν is one-to-one.

To complete the proof it is necessary only to note that since every element of G is of the form $a^{m_0}b^{m_1}a^{m_2} \dots b^{m_n}$, such a term can be re-expressed as the image of some element of G_n by merely adding and subtracting appropriate powers of b throughout.

3.14. THEOREM. *If $\{m_\alpha | \alpha \in N\}$ is any infinite subset of N , then $\bigvee \{\mathcal{S}_{m_\alpha} | \alpha \in N\} = \bigvee \{\mathcal{S}_n | n \in N\}$, i.e. there is only one variety in \mathbf{L} that is an infinite join of \mathcal{S}_n 's.*

PROOF. It is sufficient to show that for $n \in N$, $G_n \in \bigvee \{\mathcal{S}_{m_\alpha} | \alpha \in N\}$. If $n | m_\beta$ for some β , then $\mathcal{S}_n \subseteq \mathcal{S}_{m_\beta} \subseteq \bigvee \{\mathcal{S}_{m_\alpha} | \alpha \in N\}$; therefore, we may assume that for all m_α , $n \nmid m_\alpha$. We may also assume that $n < m_\alpha$ for all $\alpha \in N$, since there is only a finite set for which this is not the case, and that the set $\{\mathcal{S}_{m_\alpha}\}$ is ordered so that $m_\alpha < m_{\alpha+1}$ for all $\alpha \in N$.

Now let G be the cardinal product of the G_{m_α} 's. So elements of G are of the form $((\phi_1, k_1), (\phi_2, k_2), (\phi_3, k_3), \dots)$, where for all $\alpha \in N$, $k_\alpha \in Z$ and $\phi_\alpha: Z \rightarrow Z$ such that $\phi_\alpha(j + m_\alpha) = \phi_\alpha(j)$ for every $j \in Z$, and the k_α dominate the order.

Let us pick $a, b \in G$ of the form $a = ((\psi_1, 0), (\psi_2, 0), (\psi_3, 0), \dots)$ and $b = ((0, 1), (0, 1), (0, 1), \dots)$ where for all $\alpha \in N$, $\psi_\alpha(j)$ is 1 if $j \equiv 0 \pmod{n}$ and is 0 otherwise and $0(j) = 0$ for $0 < j < m_\alpha$ and extended mod m_α to all $j \in Z$. Then certainly $b > a > e$ in G . For $1 < i < n - 1$,

$$b^{-i}ab^i = ((\psi_{1,i}, 0), (\psi_{2,i}, 0), (\psi_{3,i}, 0), \dots),$$

where $\psi_{\alpha,i}(j) = \psi_\alpha(j - i)$ for all $\alpha \in N$ and $j \in Z$. Therefore,

$$b^{-i}ab^i \wedge a = ((\psi_{1,i} \wedge \psi_1, 0), (\psi_{2,i} \wedge \psi_2, 0), (\psi_{3,i} \wedge \psi_3, 0), \dots),$$

where $(\psi_{\alpha,i} \wedge \psi_\alpha)(j) = 0$ for all $\alpha \in N$ and $0 < j < m_\alpha - 1$ and $(\psi_{\alpha,i} \wedge \psi_\alpha)(0)$ is 1 if $i \equiv m_\alpha \pmod{n}$ and is 0 otherwise. Similarly,

$$b^{-n}ab^n = ((\psi_{1,n}, 0), (\psi_{2,n}, 0), (\psi_{3,n}, 0), \dots)$$

with $\psi_{\alpha,n}(j) = \psi_\alpha(j - n)$ for all $\alpha \in N$ and $j \in Z$. Hence,

$$b^{-n}ab^n a^{-1} = ((\psi_{1,n} - \psi_1, 0), (\psi_{2,n} - \psi_2, 0), (\psi_{3,n} - \psi_3, 0), \dots),$$

where for all $\alpha \in N$, $(\psi_{\alpha,n} - \psi_\alpha)(0) = \psi_\alpha(0 - n) - \psi_\alpha(0) = \psi_\alpha(m_\alpha - n) - \psi_\alpha(0) = 0 - 1 = -1$ since $n < m_\alpha$ and $n \nmid m_\alpha$; $(\psi_{\alpha,n} - \psi_\alpha)(j)$ is 1 if $m_\alpha \equiv (n - j) \pmod{n}$ and is 0 otherwise for $0 < j < n - 1$; and $(\psi_{\alpha,n} - \psi_\alpha)(j) = 0$ for $n < j < m_\alpha - 1$. So for any $\alpha \in N$, $(\psi_{\alpha,n} - \psi_\alpha)$ has exactly two nonzero entries, both of which occur within the first n terms.

Now, let M be the l -ideal of G generated by $b^{-n}ab^n a^{-1}$ and $\{b^{-i}ab^i \wedge a \mid 1 < i < n - 1\}$. Obviously, $b \notin M$. Also, for any $g \in M$, where $g = ((\phi_1, 0), (\phi_2, 0), (\phi_3, 0), \dots)$, there is a uniform bound on the number of nonzero entries for the ϕ_α . However, this is not true for the ψ_α , so $a \notin M$. Obviously, aM and bM in G/M satisfy the conditions of (3.13), so the l -subgroup of G/M generated by aM and bM is l -isomorphic to G_n . Thus, $G_n \in \bigvee \{\mathcal{S}_{m_\alpha} \mid \alpha \in N\}$.

Using this theorem, it is now possible to answer a question posed by Martinez [10].

3.15. COROLLARY. L is not a Brouwer lattice.

PROOF. $\mathcal{S}_3 = \mathcal{S}_3 \wedge (\bigvee \{\mathcal{S}_{2^n} \mid n \in N\})$ according to (3.14), but $\bigvee \{(\mathcal{S}_3 \wedge \mathcal{S}_{2^n}) \mid n \in N\} = \mathcal{Q}$ by (3.8).

Note that the proof above is actually of a stronger result, i.e. meets do not distribute over infinite joins even of towers in L . Also (3.14) eliminates the possibility of dropping the idea of taking quotients when generating an infinite join of varieties.

Since for all $n \in N$, G_n is an l -subgroup of $Z \text{ Wr } Z$, $\bigvee \{\mathcal{S}_n \mid n \in N\} \subseteq l\text{-var}(Z \text{ Wr } Z)$. In order to determine whether this containment is proper, it is necessary to look at a generalization of (3.13).

3.16. THEOREM. *An l -group G is l -isomorphic to $Z \wr Z$ iff it is generated (as a group) by two positive elements a and b such that*

- (i) $a < b$, and
- (ii) $b^{-i}ab^i \wedge a = e$, for all $i \in N$.

PROOF. To prove necessity, the elements $a = (\theta, 0)$ and $b = (0, 1)$ of $Z \wr Z$, where $\theta(j)$ is 1 if $j = 0$ and is 0 otherwise and $0(j) = 0$ for all $j \in Z$, can readily be shown to generate $Z \wr Z$ and satisfy conditions (i) and (ii).

For the proof of sufficiency, define $\nu: Z \wr Z \rightarrow G$ by $(\phi, k)\nu = (\prod_{i \in Z} (b^{-i}ab^i)^{\phi(i)})b^k$. Since for only a finite number of integers i is $\phi(i) \neq 0$, this function is well defined. By arguments analogous to those of (3.13), it can be shown that ν is an l -isomorphism.

3.17. THEOREM. $\bigvee \{\mathcal{S}_n | n \in N\} = l\text{-var}(Z \Wr Z) = l\text{-var}(Z \wr Z)$.

PROOF. First it should be noted that $Z \Wr Z$ is an \mathcal{L} -extension of $Z \wr Z$, i.e. every element of $Z \Wr Z$ can be expressed as a join of a pairwise disjoint subset of elements of $Z \wr Z$. As Bernau [1] has shown that l -group varieties are closed under \mathcal{L} -extensions, it suffices simply to show that $Z \wr Z \in \bigvee \{\mathcal{S}_n | n \in N\}$.

Let G be the cardinal product of the G_n . Then elements of G can be expressed in the form $((\phi_1, k_1), (\phi_2, k_2), (\phi_3, k_3), \dots)$, where for every $n \in N$, $k_n \in Z$ and $\phi_n: Z \rightarrow Z$ such that $\phi_n(j+n) = \phi_n(j)$ for all $j \in Z$, and where the k_n dominate the order in G . Let us pick $a, b \in G$ of the form $a = ((\psi_1, 0), (\psi_2, 0), (\psi_3, 0), \dots)$ and $b = ((0, 1), (0, 1), \dots)$ where for all $n \in N$, $\psi_n(j)$ is 1 if $j \equiv 0 \pmod{n}$ and is 0 otherwise and $0(j) = 0$ for all $j \in Z$. Thus, $b > a > e$.

For any $i \in N$, $b^{-i}ab^i = ((\psi_{1,i}, 0), (\psi_{2,i}, 0), (\psi_{3,i}, 0), \dots)$ where $\psi_{n,i}(j) = \psi_n(j-i)$ for all $n \in N$ and $j \in Z$. Therefore,

$$b^{-i}ab^i \wedge a = ((\psi_{1,i} \wedge \psi_1, 0), (\psi_{2,i} \wedge \psi_2, 0), (\psi_{3,i} \wedge \psi_3, 0), \dots)$$

where for any $n \in N$, $(\psi_{n,i} \wedge \psi_n)(j) = 1$ if both $i, j \equiv 0 \pmod{n}$ and is 0 otherwise. So for any given i , only a finite number of n 's will divide i ; thus, only a finite number of the $(\psi_{n,i} \wedge \psi_n)$ will be nonzero functions for that i .

Let M be the l -ideal of G generated by $\{b^{-i}ab^i \wedge a | i \in N\}$. Obviously, $b \notin M$; and since all of the ψ_n are nonzero functions, $a \notin M$. Hence the l -subgroup of G/M generated by aM and bM is l -isomorphic to $Z \wr Z$ by (3.16), and $Z \wr Z \in \bigvee \{\mathcal{S}_n | n \in N\}$.

4. Varieties and torsion classes. An alternate system of classification of l -groups is that of torsion classes [11]. A *torsion class* is a collection of l -groups that is closed under taking convex l -subgroups, l -homomorphic images, and joins of convex l -subgroups in the collection. Let \mathbf{T} denote the class of all torsion classes of l -groups. Then using definitions analogous to those for varieties, it is possible to define the operations of meet, join, and multiplication on \mathbf{T} . It is also worthy of note that for any l -group G , the join of all convex l -subgroups of G belonging to a torsion class \mathfrak{T} is itself in \mathfrak{T} . This join is called the \mathfrak{T} -torsion radical of G , denoted $\mathfrak{T}(G)$, and is a characteristic l -ideal of G . It follows that $G \in \mathfrak{T}$ iff $G = \mathfrak{T}(G)$, and $G \in \mathfrak{T} \vee \mathfrak{U}$ iff $G = \mathfrak{T}(G) + \mathfrak{U}(G)$ for any $\mathfrak{T}, \mathfrak{U} \in \mathbf{T}$.

Holland has shown that all l -group varieties are torsion classes [7], and Martinez has shown that \mathbf{T} is a complete Brouwer lattice [11]. Hence, by (3.15) we have that \mathbf{L} is not a complete sublattice of \mathbf{T} , as otherwise it would inherit the Brouwerian structure. It then remains to ask whether \mathbf{L} is even a sublattice of \mathbf{T} .

4.1. DEFINITION. A variety \mathcal{V} is said to have the *lex property* if for every $G \in \mathcal{V}$, $G_{\#} \in \mathcal{V}$, where $G_{\#} = G \times Z$.

4.2. PROPOSITION. If $\{\mathcal{V}_{\alpha} | \alpha \in \mathfrak{A}\}$ is a collection of varieties having the lex property and $\mathcal{U} \in \mathbf{L}$, then:

- (i) $\bigwedge \mathcal{V}_{\alpha}$ and $\bigvee \mathcal{V}_{\alpha}$ have the lex property,
- (ii) $\mathcal{U} \mathcal{V}_{\alpha}$ has the lex property.

PROOF. (i) That $\bigwedge \mathcal{V}_{\alpha}$ has the lex property is obvious. If $G \in \bigvee \mathcal{V}_{\alpha}$, then $G \approx H/K$ where H has l -ideals N_{α} with $\bigcap N_{\alpha} = \{e\}$ and $H/N_{\alpha} \in \mathcal{V}_{\alpha}$ for all $\alpha \in \mathfrak{A}$. Then $G_{\#} \approx (H/K)_{\#} \approx H_{\#}/(K \times \{0\})$, where $N_{\alpha} \times \{0\}$ is an l -ideal of $H_{\#}$ with $\bigcap (N_{\alpha} \times \{0\}) = \{(e, 0)\}$ and $H_{\#}/(N_{\alpha} \times \{0\}) \approx (H/N_{\alpha})_{\#} \in \mathcal{V}_{\alpha}$, since \mathcal{V}_{α} has the lex property. Thus, $G_{\#} \in \bigvee \mathcal{V}_{\alpha}$.

(ii) If $G \in \mathcal{U} \mathcal{V}_{\alpha}$, then G has an l -ideal $N \in \mathcal{U}$ with $G/N \in \mathcal{V}_{\alpha}$. Therefore, $N \times \{0\}$ is an l -ideal of $G_{\#}$ with $G_{\#}/(N \times \{0\}) \approx (G/N)_{\#} \in \mathcal{V}_{\alpha}$ since \mathcal{V}_{α} has the lex property. Hence, $G_{\#} \in \mathcal{U} \mathcal{V}_{\alpha}$.

4.3. THEOREM. Let $\{\mathcal{V}_{\alpha} | \alpha \in \mathfrak{A}\}$ be a collection of varieties, all having the lex property, such that $\mathcal{V}_{\beta} \subset_{\mathbf{L}} \bigvee \{\mathcal{V}_{\alpha} | \alpha \in \mathfrak{A}\}$ for all $\beta \in \mathfrak{A}$. Then $_{\mathbf{L}} \bigvee \mathcal{V}_{\alpha} \supset _{\mathbf{T}} \bigvee \mathcal{V}_{\alpha}$ (where $_{\mathbf{L}} \bigvee$ and $_{\mathbf{T}} \bigvee$ denote joins in \mathbf{L} and \mathbf{T} respectively).

PROOF. Since $\mathbf{L} \subseteq \mathbf{T}$, $_{\mathbf{L}} \bigvee \mathcal{V}_{\alpha} \supseteq _{\mathbf{T}} \bigvee \mathcal{V}_{\alpha}$. Thus, we need only show the containment to be proper. Without loss of generality, we may assume that \mathfrak{A} is well-ordered and that for any $\alpha \in \mathfrak{A}$, $\mathcal{V}_{\alpha} \not\subseteq \bigcup \{\mathcal{V}_{\beta} | \beta < \alpha\}$; since otherwise, such “unnecessary” varieties could be thrown out without affecting the final join in either \mathbf{L} or \mathbf{T} .

If \mathfrak{A} has no largest element, then choose $G_{\alpha} \in \mathcal{V}_{\alpha} \setminus \bigcup \{\mathcal{V}_{\beta} | \beta < \alpha\}$, for each $\alpha \in \mathfrak{A}$. Then by (4.2), $G = (\prod G_{\alpha})_{\#} \in _{\mathbf{L}} \bigvee \mathcal{V}_{\alpha}$. If $G \in _{\mathbf{T}} \bigvee \mathcal{V}_{\alpha}$, then $G = \sum \mathcal{V}_{\alpha}(G)$, where $\mathcal{V}_{\alpha}(G)$ is the \mathcal{V}_{α} -torsion radical of G and hence an l -ideal of G . All proper l -ideals of G are contained within $(\prod G_{\alpha}) \times \{0\}$, and such l -ideals cannot generate G . Thus, for some $\beta \in \mathfrak{A}$, $G = \mathcal{V}_{\beta}(G)$, implying that $G \in \mathcal{V}_{\beta}$. Since $G_{\beta+1}$ is an l -subgroup of G chosen outside of \mathcal{V}_{β} , the desired contradiction has been reached.

If \mathfrak{A} has a largest element α_0 , then for some $\gamma \in \mathfrak{A}$, $\mathcal{V}_{\gamma} \not\subseteq \mathcal{V}_{\alpha_0}$. So, choose $G_{\alpha} \in \mathcal{V}_{\alpha} \setminus \bigcup \{\mathcal{V}_{\beta} | \beta < \alpha\}$ for each $\alpha \in \mathfrak{A}$ as before, and choose $H_{\gamma} \in \mathcal{V}_{\gamma} \setminus \mathcal{V}_{\alpha_0}$. Then $G = (H_{\gamma} \times \prod G_{\alpha})_{\#} \in _{\mathbf{L}} \bigvee \mathcal{V}_{\alpha}$, and by an argument similar to that above, $G \notin _{\mathbf{T}} \bigvee \mathcal{V}_{\alpha}$.

In order to state finally that \mathbf{L} is not a sublattice of \mathbf{T} , it remains to be determined whether or not there exist varieties having the lex property, and if some exist, how prevalent they are in \mathbf{L} . First, as a lex product is simply a direct product of the groups involved, any l -group variety definable solely in terms of its group operations will have the lex property. Thus, \mathcal{Q} and \mathcal{L}_n ($n \in \mathbf{N}$) have the lex property. In fact, all of the varieties to which reference has been made within this paper have the lex property, making \mathbf{L} very far indeed from being a sublattice of \mathbf{T} .

4.4. PROPOSITION. \mathcal{U} , \mathcal{R} , \mathcal{W} (the weakly abelian variety), and \mathcal{S}_n ($n \in N$) have the lex property.

PROOF. (i) For $G \in \mathcal{U}$, we wish to show that $x_1^2 x_2^2 > x_2 x_1$ for all positive elements $x_1, x_2 \in G_\#$. Let $x_1 = (g, k)$ and $x_2 = (h, l)$ where $g, h \in G$ and $k, l \in \mathbb{Z}$. These elements are positive iff $k > 0$ or $k = 0$ and $g \in G^+$ and $l > 0$ or $l = 0$ and $h \in G^+$. If either $k > 0$ or $l > 0$, then $2k + 2l > l + k$, making

$$(g, k)^2(h, l)^2 = (g^2 h^2, 2k + 2l) > (hg, l + k) = (h, l)(g, k).$$

If both $k, l = 0$, then $(g, 0)^2(h, 0)^2 = (g^2 h^2, 0) > (hg, 0) = (h, 0)(g, 0)$ since $G \in \mathcal{U}$. Thus, $G_\# \in \mathcal{U}$.

(ii) For $G \in \mathcal{R}$, we wish to show that $x_1 \wedge (x_2^{-1} x_1^{-1} x_2) < e$ for all $x_1, x_2 \in G_\#$. Again letting $x_1 = (g, k)$ and $x_2 = (h, l)$, we have that

$$(g, k) \wedge (h, l)^{-1}(g, k)^{-1}(h, l) = (g, k) \wedge (h^{-1} g^{-1} h, -k),$$

which equals (g, k) if $k < 0$ and $(g \wedge h^{-1} g^{-1} h, 0)$ if $k = 0$ and $(h^{-1} g^{-1} h, -k)$ if $k > 0$. In all cases, the outcome is less than or equal to $(e, 0)$; so $G_\# \in \mathcal{R}$.

(iii) For $G \in \mathcal{W}$, it is sufficient to show that $x_1^2 > x_2^{-1} x_1 x_2$ for any x_2 and positive x_1 in $G_\#$. Letting $x_1 = (g, k)$ and $x_2 = (h, l)$, with either $k > 0$ or $k = 0$ and $g \in G^+$, then clearly $(g, k)^2 = (g^2, 2k) > (h^{-1} g h, k) = (h, l)^{-1}(g, k)(h, l)$. So $G_\# \in \mathcal{W}$.

(iv) To show that \mathcal{S}_n has the lex property, it suffices to show that $(G_n)_\# \in \mathcal{S}_n$. Let $G = (\prod_{i=1}^\infty G_n) / (\sum_{i=1}^\infty G_n)$; then $G \in \mathcal{S}_n$. Let H_n be the diagonal l -subgroup of $\prod_{i=1}^\infty G_n$, with K_n its canonical image in G . Clearly K_n and G_n are l -isomorphic; let us denote this l -isomorphism by ν . Let $g \in G$ correspond to $((0, n), (0, 2n), (0, 3n), \dots) \in \prod_{i=1}^\infty G_n$. Then for any $h \in K_n$, $g > h$ and $gh = hg$. Hence, the l -subgroup of G generated by K_n and g is l -isomorphic to $(G_n)_\#$ via the map which sends hg^m to $(h\nu, m)$ for $h \in K_n$ and $m \in \mathbb{Z}$. So $(G_n)_\# \in \mathcal{S}_n$.

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